HIGHER COHERENCE AND A GENERALIZATION OF HIGHER CATEGORIFIED ALGEBRAIC STRUCTURES

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1. Introduction

Discovery or recognition
of the right kind of algebraic structure
is often important
in the development of mathematical subjects.
Starting perhaps with "group" in Galois theory,
a list of examples would easily get long.
Can we systematically
find and treat **algebraic structures**?

Question seems important
for the use of **higher categorical** ideas.
With high categorical dimensionality,
more variety of structures available.
Why higher category theory?

- Necessary for both analysis and construction of **topological field theories**, as has become clear
from the solution and generalization
of Baez and Dolan's **cobordism hypothesis** by Lurie and Hopkins.
Lurie argues that an instance
of CH proved earlier was a theorem
by Costello, which was for proposing a definition of the **B-model** in the mirror symmetry.

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- **Categorification** has been a useful method
  for finding important new structures (since Grothendieck).

We apply a concrete understanding of **higher coherence**, and find something systematic at a quite global level.

In order to proceed, we need a few reminders.

Coloured operad/**Multicategory**, a generalization
of symmetric monoidal category:

For $\mathcal{A}$ symmetric monoidal category,
underlying multicategory $\Theta\mathcal{A}$, from which
$\mathcal{A}$ can be recovered.

A multicategory **controls algebras** *over it...*

For $\mathcal{U}$ a multicategory

$\mathcal{U}$-**algebra** in a symmetric monoidal category $\mathcal{A}$
= functor $\mathcal{U} \to \Theta\mathcal{A}$ of multicategories.

Multicategory is analogous to an **algebraic theory** in the sense of **Lawvere**.

Many kinds of algebraic structure (in a symmetric monoidal category)
controlled by a multicategory.

Commutative: by "Com", terminal.

Associative: by "E_1".

Bare object (no structure): by "Init", initial uncoloured operad.

...and so on.

With colours more variety of structures can be controlled.

Example of **categorified** structure:

For $\mathcal{U}$ a multicategory

"$\mathcal{U}$-monoidal category" — $\mathcal{U}$-algebra in categories.

Symmetric monoidal, associative monoidal, braided, ...

Back to the subject...

When the notion "X-algebra" (e.g., $\mathcal{U}$-algebra, multicategory, etc.) has a categorification, "categorified X-algebra" sometimes has

a good generalization, "**X-theory**".

The meaning of "good" later.

We call the notion of X-theory a **theorization** of the notion of X-algebra.

"Multicategory" is a theorization
of "commutative algebra", which
generalizes "symmetric monoidal category".

There is a natural

theorization "$\mathcal{U}$-**graded** multicategory" of "$\mathcal{U}$-algebra",
generalizing "$\mathcal{U}$-monoidal category".

$$E_1\text{-graded} = \text{planar}$$

$$E_2\text{-graded} = \text{braided}$$

$$\text{Init-graded multicategory} = \text{category}$$

$\mathrm{Multicat}_{\mathcal{U}}(\mathrm{Set}) = \mathrm{Multicat}(\mathrm{Set})_{/\mathcal{U}}$.
In the case $\mathcal{U} \in \mathrm{Multicat}(\mathrm{Gpd})$, e.g., $\mathcal{U} = E_2$,

$$\mathrm{Multicat}_{\mathcal{U}}(\mathrm{Gpd}) = \mathrm{Multicat}(\mathrm{Gpd})_{/\mathcal{U}}.$$

Example: For $\mathcal{C}$ a category

= multicategory with only **unary** multimaps

Algebra: Functor on $\mathcal{C}$ ("left $\mathcal{C}$-module").

Categorification: Functor $\mathcal{C} \to \mathrm{Cat}$.

Theorization: In Set,

category $\mathcal{X}$ + functor $\mathcal{X} \to \mathcal{C}$,

among which,

categorifications (and their lax morphisms) correspond to **fibrations**

$$\begin{array}{c}\mathcal{X}^{\mathrm{op}}\\ \downarrow\\ \mathcal{C}^{\mathrm{op}}\end{array}$$

(and not necessarily Cartesian functors over $\mathcal{C}^{\mathrm{op}}$).

"($\mathcal{U}$-graded) 2-**theory**" — theorization of "($\mathcal{U}$-graded) multicategory" (= "1-**theory**").

Example: For $\mathcal{M}$ a planar multicategory,
"Morita" Init-graded 2-theory $\mathcal{A}lg_1(\mathcal{M})$, where

$$\mathcal{A}lg_1(\Theta\mathcal{A}) \simeq \Theta\mathsf{Alg}_1(\mathcal{A})$$

for $\mathcal{A}$ a associative monoidal category with nice behaviour,
$\mathsf{Alg}_1(\mathcal{A})$ "Morita" 2-category (= categorified Init-graded 1-theory) due to Bénabou.

$\mathcal{A}lg_1$ will appear again later.

2. Coherence and higher theories

Concretely, can theorize
by using an inductivity
embedded in the structure of the **coherence** for higher associativity:

Consider situation:

m**:** system of operations
wanting to be **associative**, operating
as maps in an symmeric monoidal $(\infty, 1)$-category.
E.g., "m + coherent associativity" may
define an "X-algebra".

m'**:** system of (2-)isomorphisms/homotopies
giving an associativity of m.

m' wants to be **coherent**.

In this situation,
see m' **as operations** themselves.

Invertibility of those operations **not** required for a (op)**lax** X-algebra.

Then

Coherence *of the associativity* m' *for* m
= *Coherent* **associativity** *of* m' **as operations**

Idea for theorizing: If m
gives an X-algebra structure,
then "X-theory" is a kind
defined by **operations** m'

...considered in the "categorical deloop" $B\mathcal{A}$, so 2-morphisms m' are **maps** in $\mathcal{A} = \mathsf{End}_{B\mathcal{A}}(*)$.

Conceptual significance of theorization to be discussed later.

We can iterate theorization,
get to "n-**theory**", generalizing n-**categories** in particular.
Definition of an n-theory can be written explicitly.

There are general constructions
of higher theories, which play various roles.

- "Delooping" construction $\mathbb{B}$, of an $(n+1)$-theory from an n-theory.
  For $\mathcal{A}$ a symmetric monoidal category (= categorified 0-theory)
  $$\Theta^{n+1} B^n \mathcal{A} \simeq \mathbb{B}^n \Theta \mathcal{A},$$
  equivalence of $(n+1)$-theories.
- Change of "grading": pull-back,
  push-forward on left and on right.
- Day convolution.

Many of these **raises** theoretic order.
If at least interested
in multicategories, then
would want to know all
higher theories.

Also concrete constructions
in more specific situations.

3. Meaning of theorization

Relevance of categorification for us:

"X-algebra" makes sense
in a categorified form
of X-algebra.

E.g.,

$\mathcal{U}$-algebra in a $\mathcal{U}$-**monoidal** category $\mathcal{A}$
$= \text{lax } \mathcal{U}\text{-monoidal functor } \mathbf{1}^0_{\mathcal{U}} \to \mathcal{A}$,

$\mathbf{1}^0_{\mathcal{U}}$ unit $\mathcal{U}$-monoidal.
...an **enriched** notion
of $\mathcal{U}$-algebra.

"$\mathcal{U}$-**graded** multicategory"

was a common generalization

of "$\mathcal{U}$-monoidal category" and "Multicategory".

Notion of $\mathcal{U}$-algebra can be **enriched** in a $\mathcal{U}$-graded multicategory.

In general, by theorizing "X-algebra",

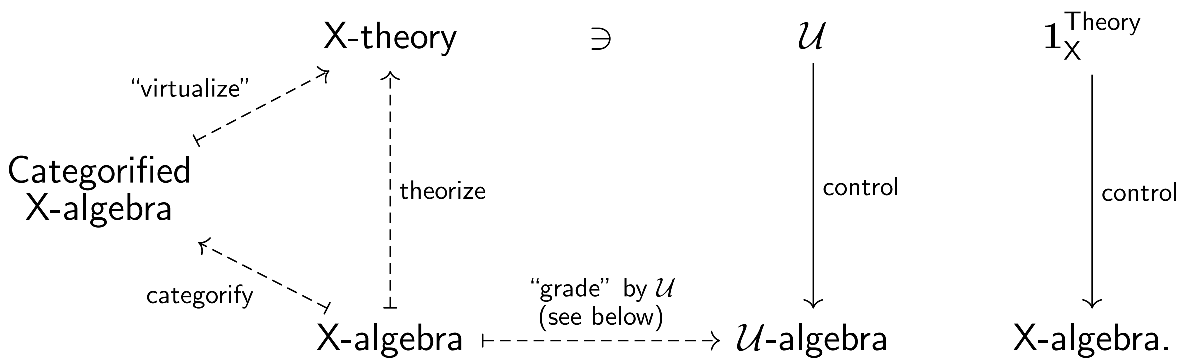
we want...

"For $\mathcal{A}$ a categorified form of X-algebra

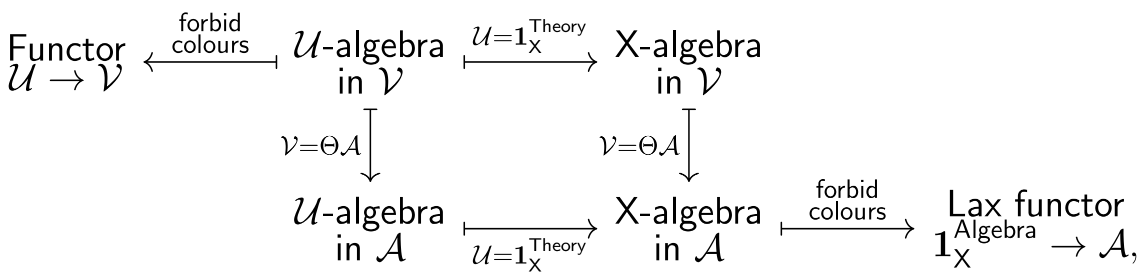
X-algebra in $\mathcal{A}$

$=$ *"coloured" functor* $\mathbf{1}_{\mathsf{X}}^{\mathsf{Theory}} \to \Theta\mathcal{A}$ *of X-theories*"

- $\mathbf{1}_{\mathsf{X}}^{\mathsf{Theory}}$ terminal X-theory,
- $\Theta\mathcal{A}$ denotes $\mathcal{A}$ **as** an X-theory.
- Left hand side = "coloured" lax functor $\mathbf{1}_{X}^{\mathsf{Algebra}} \to \mathcal{A}$ of X-algebras.



Enriched notions



where

$\mathcal{U}$, $\mathcal{V}$: X-theories

$\mathcal{A}$: categorified X-algebra

4. Graded higher theories

Are graded multicategories controlled? By what?

— By a very **simple** 2-theory.

In comparison,

classical answer: "By a multicategory".

E.g., "slice" multicategory $\mathcal{U}^+$ of **Baez** and **Dolan**

$$\mathsf{Alg}_{\mathcal{U}^+}(\mathsf{Set}) = \mathsf{Multicat}_{\text{Same colours as } \mathcal{U}\text{'s}}(\mathsf{Set})_{/\mathcal{U}}.$$

But a **simpler** answer above by:

expressing "1-dimensional" structure (multicategory)

as algebra over a 2-**dimensional** structure.

Purpose of Baez and Dolan was to define n-category.

Iterated theorization is a more direct route to n-categories

compared to Baez and Dolan.

Key: an n-category is naturally n-**dimensional** as is an n-theory,

rather than 1-dimensional like a multicategory.

Also helps with **enriching** the notion...

Where to enrich "$\mathcal{U}$-graded multicategory"?

— In a $(\mathcal{U}\otimes E_1)$-monoidal category (folklore? ...if $\mathcal{U}$-graded multicategories

in a **symmetric** monoidal category had been known

as seems possible.)

$(E_1 \otimes E_1 = E_2$ etc.),

and more generally, ...

Let us see the details.

For $\mathcal{U}$ an n-theory, "$\mathcal{U}$-**graded** n-**theory**", theorizing "$\mathcal{U}$-algebra".

Theorem: *Following notions are equivalent:*

- *$\mathcal{U}$-graded n-theory.*
- *$\Theta\mathcal{U}$-algebra.*

$n = 1$: *$\mathcal{U}$-graded multicategories are controlled*

by the 2-theory $\Theta\mathcal{U}$

$$\mathsf{Multicat}_{\mathcal{U}} = \mathsf{Alg}_{\Theta\mathcal{U}}.$$

$\Theta\mathcal{U}$ is simple and direct

...compared to $\mathcal{U}^+$, which was the main construction

for Baez and Dolan.

Construction of $\Theta\mathcal{U}$ is in one step:

Replace composition operation in $\mathcal{U}$

with the corepresented "bimodule".

...essentially as simple as the underlying multicategory of a symmetric monoidal category.

"$\mathcal{U}$-graded n-theory" has a natural theorization, and

Theorem: *Following notions are equivalent:*

- *$\mathcal{U}$-graded $(n+1)$-theory.*
- *$\Theta\mathcal{U}$-graded $(n+1)$-theory.*

Leads to iterative theorization.

For $\mathcal{U}$ a multicategory $=$ 1-theory,
a $\mathcal{U}$-graded 2-theory
is a general place where
the notion of $\mathcal{U}$-graded multicategory
can be enriched,
specializing to enrichment
in a $(\mathcal{U} \otimes E_1)$-monoidal category.

Example: $\mathcal{U} =$ Init,
$\mathcal{M}$ a planar mutlicategory.
There is forgetful functor

$$\mathcal{A}lg_1(\mathcal{M}) \longrightarrow \mathbb{B}\mathcal{M}$$

of Init-graded 2-theories,
generalizing the forgetful lax functor

$$\mathrm{Alg}_1(\mathcal{A}) \longrightarrow B\mathcal{A}$$

for $\mathcal{A}$ a nice associative monoidal category.
A category $=$ Init-graded 1-theory, enriched
in $\mathcal{M}$ or "along" $\mathbb{B}\mathcal{M}$,
has a canonical lift
to a category enriched
along $\mathcal{A}lg_1(\mathcal{M})$.

5. Graded lower theories

$\mathcal{U}$ an n-theory.
For $0 \le m \le n-1$, there is a notion of $\mathcal{U}$-**graded** m-**theory** so that

$\mathcal{U}$-algebra $=$ $\mathcal{U}$-graded $(n-1)$-theory.

Moreover, a notion of "ℓ-theory" graded by a $\mathcal{U}$-**graded** m-theory.
E.g.,

$\mathbf{1}^m_{\mathcal{U}}$-graded ℓ-theory $=$ $\mathcal{U}$-graded ℓ-theory

6. More general higher theories

Can also theorize some structures

involving operations with multiple inputs **and** multiple outputs,

e.g.,

"coloured properad" of **Vallette**, which turns out

to be "Cocorr(Fin)-graded" 1-theory,

Cocorr(Fin) cocorrespondence category on finite sets.

We get iterative theorizations.

For $\mathcal{C}$ a category,

a "$\mathrm{Bord}_1^{\mathrm{oriented}}$-graded" 1-theory $\mathcal{Z}_{\mathcal{C}}$ such that

every 1-dimensional "TFT" **in** $\mathcal{Z}_{\mathcal{C}}$

$$\mathbf{1}^1_{\mathrm{Bord}_1} \longrightarrow \mathcal{Z}_{\mathcal{C}}$$

is **of the form**

$$\mathbf{1}^1_{\mathrm{Bord}_1} = \mathcal{Z}_{\mathbf{1}} \xrightarrow{\mathcal{Z}_x} \mathcal{Z}_{\mathcal{C}}$$

for an unique object

$$x\colon \mathbf{1} \longrightarrow \mathcal{C}$$

of $\mathcal{C}$.

Very different from field theories in **untheorized** context,

classified by **dualizable** objects of a **symmetric monoidal** category.